Coherence Properties of Blackbody Radiation.* I. Correlation Tensors of the Classical Field

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This paper is concerned with the extension of some recently reported results, especially those of Bourret (1960), relating to coherence properties of blackbody radiation. An explicit expression for the complex electric correlation tensor $\mathcal{E}_{ij}(\mathbf{r},\tau)$ of blackbody radiation is derived, and on the basis of it spatial coherence ($\tau=0$) is discussed in detail. The behavior of diagonal as well as nondiagonal components is illustrated by contour diagrams. In particular, it is found that the nondiagonal components of the correlation tensor, even though being zero for zero-space separation ($\mathbf{r}=0$), acquire, in general, nonvanishing values when $\mathbf{r}\neq 0$. The magnetic and mixed correlation tensors are also discussed.

1. INTRODUCTION

It is a common notion that the most incoherent radiation is blackbody radiation in an equilibrium enclosure. Recent researches[1–4] have shown that even in this type of radiation there is coherence in a sufficiently small space-time region. Bourret[1] has derived expressions for the second-order electric correlation tensor of blackbody radiation, using techniques analogous to those employed in the theory of isotropic turbulence of an incompressible fluid. Such a correlation tensor describes the correlation between the Cartesian components (denoted by subscripts i, j) of the electric field $\mathbf{E}^{(r)}(\mathbf{r},t)$ at two space-time points $\mathbf{r}_1$ and $\mathbf{r}_2$, at time instants t_1 and t_2,

$$\mathcal{E}_{ij}^{(r)}(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2)=\langle E_i^{(r)}(\mathbf{r}_1,t_1)E_j^{(r)}(\mathbf{r}_2,t_2)\rangle, \quad (1.1)$$

where sharp brackets denote the time average:

$$\langle E_i^{(r)}(\mathbf{r}_1,t_1)E_j^{(r)}(\mathbf{r}_2,t_2)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E_i^{(r)}(\mathbf{r}_1,t_1+t)E_j^{(r)}(\mathbf{r}_2,t_2+t)dt. \quad (1.2)$$

Sarfatt[3] rederived some of Bourret's results using quantum mechanical density matrix techniques.

The components of the correlation tensors discussed by Bourret and Sarfatt are real functions of space and time. However, numerous recent researches on coherence properties of light have shown that an appropriate measure of coherence is provided by certain complex rather than real correlation functions (cf. Ref. 5), and this is also supported by recent investigations of Glauber[6] based on quantum field theoretical calculations. (See also Sudarshan,[7a,b] Wolf,[8] Mandel, Sudarshan and Wolf,[9] Mandel.[10]) The appropriate complex electric correlation tensor $\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2)$ may be derived from the real correlation tensor defined by (1.1), by using the concept of an analytic signal.[11,5] Assuming the field to be stationary in time, so that $\mathcal{E}_{ij}^{(r)}$ depends on t_1 and t_2 only through the time difference $\tau=t_1-t_2$, the complex correlation tensor is given by[12]

$$\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=2[\mathcal{E}_{ij}^{(r)}(\mathbf{r}_1,\mathbf{r}_2,\tau)+i\mathcal{E}_{ij}^{(i)}(\mathbf{r}_1,\mathbf{r}_2,\tau)], \quad (1.3)$$

where

$$\mathcal{E}_{ij}^{(i)}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathcal{E}_{ij}^{(r)}(\mathbf{r}_1,\mathbf{r}_2,\tau')}{\tau'-\tau}d\tau' \quad (1.4)$$

is the Hilbert transform of $\mathcal{E}_{ij}^{(r)}(\mathbf{r}_1,\mathbf{r}_2,\tau)$ and P denotes the Cauchy principal value at $\tau'=\tau$.

Bourret and Sarfatt have restricted their discussion of the tensor to certain special cases only, namely, to those characterizing temporal correlation ($\mathbf{r}_1=\mathbf{r}_2$) and lateral and longitudinal spatial correlation ($\tau=0$). Their results do not provide information about the nondiagonal components of the correlation tensor, though, in principle such information could be obtained with the help of their formulas.

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[1] R. C. Bourret, Nuovo Cimento **18**, 347 (1960).

[2] Y. Kano and E. Wolf, Proc. Phys. Soc. (London) **80**, 1273 (1962).

[3] J. Sarfatt, Nuovo Cimento **27**, 1119 (1963).

[4] C. L. Mehta, Nuovo Cimento **28**, 401 (1963).

[5] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Inc., New York, 1959), Chap. X.

[6] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).

[7] E. C. G. Sudarshan, (a) Phys. Rev. Letters **10**, 277 (1963); (b) in *Proceedings of the Symposium on Optical Masers* (Polytechnic Press, Brooklyn, New York and J. Wiley & Sons, Inc., New York, 1963), p. 45.

[8] E. Wolf, in *Proceedings of the Symposium on Optical Masers* (Polytechnic Press, Brooklyn, New York, and John Wiley & Sons Inc., New York, 1963), p. 29.

[9] L. Mandel, E. C. G. Sudarshan, and E. Wolf (to be published).

[10] L. Mandel, Phys. Letters **7**, 117 (1963).

[11] D. Gabor, J. Inst. Electr. Engrs. **93**, Part III, 429 (1946).

[12] An alternative, but equivalent, definition of the complex correlation tensor, which will be needed later is as follows: With the real field $\mathbf{E}^{(r)}(\mathbf{r},t)$ we associate the complex analytic signal $\mathbf{E}(\mathbf{r},t)$, i.e.,

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}^{(r)}(\mathbf{r},t)+i\mathbf{E}^{(i)}(\mathbf{r},t),$$

where $\mathbf{E}^{(i)}$ is the Hilbert transform of $\mathbf{E}^{(r)}$. Then the complex correlation tensor $\mathcal{E}_{ij}$ may be expressed in the form

$$\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\langle E_i(\mathbf{r}_1,t+\tau)E_j^*(\mathbf{r}_2,t)\rangle. \quad (1.3a)$$

The equivalence of the definitions (1.3) and (1.3a) is shown in Ref. 17, pp. 464–466, where also a certain mathematical refinement connected with questions of convergence is discussed.

In this paper an explicit expression for the complete second-order complex electric correlation tensor of blackbody radiation is derived and a number of diagrams are obtained, which show the behavior of the elements of this tensor. The magnetic and the mixed second-order complex correlation tensors are also discussed.

In paper II (Ref. 13) the electromagnetic correlation tensors defined on the basis of quantum field theory will be evaluated for blackbody radiation and will be compared with the corresponding tensors defined in the present paper on the basis of semiclassical theory. Higher order correlation tensors will also be considered in paper II.

2. EXPLICIT EXPRESSION FOR THE SECOND-ORDER COMPLEX ELECTRIC CORRELATION TENSOR OF BLACKBODY RADIATION

In this section a new expression for the normalized complex electric correlation tensor $\gamma_{ij}(\mathbf{r},\tau)$ of blackbody radiation will be derived. For any stationary field, the normalized correlation tensor is defined by the formula

$$\gamma_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\frac{\mathcal{E}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)}{[\mathcal{E}_{ii}(\mathbf{r}_1,\mathbf{r}_1,0)]^{1/2}[\mathcal{E}_{jj}(\mathbf{r}_2,\mathbf{r}_2,0)]^{1/2}}, \quad (2.1)$$

where $\mathcal{E}_{ij}$ is the correlation function (1.3). The normalization ensures[5] that $0\leq|\gamma_{ij}|\leq 1$. In the case of blackbody radiation in a cavity whose linear dimensions are large compared with the mean wavelength of the radiation, the field is isotropic and $\mathcal{E}_{ij}$ and γ_{ij} depend effectively on $\mathbf{r}_1$ and $\mathbf{r}_2$ through the difference $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ only.

An integral expression for $\gamma_{ij}(\mathbf{r},\tau)$ for blackbody radiation has been obtained by Kano and Wolf.[2] They showed that

$$\gamma_{ij}(\mathbf{r},\tau)=\frac{45\alpha^4}{8\pi^5}\int\frac{k^2\delta_{ij}-k_ik_j}{k\{\exp(\alpha k)-1\}}\times\exp\{i(\mathbf{k}\cdot\mathbf{r}-kc\tau)\}d^3k, \quad (2.2)$$

where

$$\alpha=\hbar c/KT, \quad (2.3)$$

c being the vacuum velocity of light, $\hbar$ the Planck's constant divided by 2π, K the Boltzmann constant and T the absolute temperature. The integration in (2.2) is taken over the whole $\mathbf{k}$ space. The normalization constant in (2.1) has the value

$$[\mathcal{E}_{ii}(\mathbf{r}_1,\mathbf{r}_1,0)]^{1/2}[\mathcal{E}_{jj}(\mathbf{r}_1,\mathbf{r}_2,0)]^{1/2}\equiv\mathcal{E}_{ii}(0,0,0) \quad \text{(no summation)}$$

$$=\frac{64}{45}\frac{\pi^6K^4T^4}{(hc)^3}. \quad (2.4)$$

Equation (2.2) may be written as

$$\gamma_{ij}(\mathbf{r},\tau)=\frac{45\alpha^4}{8\pi^5}(\partial_i\partial_j-\delta_{ij}\nabla^2)\int\frac{\exp\{i(\mathbf{k}\cdot\mathbf{r}-kc\tau)\}}{k(e^{\alpha k}-1)}d^3k, \quad (2.5)$$

where $\partial_i\equiv\partial/\partial r_i$. Using spherical polar coordinates for $\mathbf{k}$, with the polar axis along the direction of $\mathbf{r}$, we obtain

$$\gamma_{ij}(\mathbf{r},\tau)=\frac{45\alpha^4}{8\pi^5}(\partial_i\partial_j-\delta_{ij}\nabla^2)\int_0^\infty kdk\frac{e^{-ikc\tau}}{e^{\alpha k}-1}\int_0^\pi e^{ikr\cos\theta}\sin\theta d\theta\int_0^{2\pi}d\phi$$

$$=\frac{45\alpha^4}{2\pi^4}(\partial_i\partial_j-\delta_{ij}\nabla^2)\frac{1}{r}\int_0^\infty\frac{\sin kr\, e^{-ikc\tau}}{e^{\alpha k}-1}dk. \quad (2.6)$$

The last integral can be written as

$$\sum_{n=1}^\infty\frac{1}{2i}\int_0^\infty\{e^{ik(r-c\tau)}-e^{-ik(r+c\tau)}\}e^{-n\alpha k}dk=\frac{1}{2i}\sum_{n=1}^\infty\left\{\frac{1}{n\alpha-i(r-c\tau)}-\frac{1}{n\alpha+i(r+c\tau)}\right\},$$

so that γ_{ij} may be expressed in the form

$$\gamma_{ij}(\mathbf{r},\tau)=\frac{45\alpha^4}{2\pi^4}(\partial_i\partial_j-\delta_{ij}\nabla^2)\sum_{n=1}^\infty\frac{1}{(n\alpha+ic\tau)^2+r^2}.$$

Carrying out the differentiation on the right-hand side, we finally obtain the following expression for the normalized complex electric correlation tensor of blackbody radiation:

$$\gamma_{ij}(\mathbf{r},\tau)=\frac{90\alpha^4}{\pi^4}\sum_{n=1}^\infty\left\{\frac{\delta_{ij}}{\{(n\alpha+ic\tau)^2+r^2\}^2}+2\frac{r_ir_j-r^2\delta_{ij}}{\{(n\alpha+ic\tau)^2+r^2\}^3}\right\}, \quad (2.7)$$

where r_i, r_j are the components of the vector $\mathbf{r}$ with respect to i and j axes, respectively.

3. TEMPORAL COHERENCE

Restricting ourselves first to the case $\mathbf{r}=0$, Eq. (2.7) gives the following expression for the normalized tensor that characterizes *temporal coherence*:

$$\gamma_{ij}(0,\tau)=(90/\pi^4)\zeta(4,1+ic\tau/\alpha)\delta_{ij}, \quad (3.1)$$

where $\zeta(s,a)$ is the generalized Riemann zeta function[14]

[13] C. L. Mehta and E. Wolf, following paper, Phys. Rev. **134**, A1149 (1964).

[14] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1958), p. 265; (also Dover Publications, Inc., New York, 1962), p. 265.

defined by

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}. \quad (3.2)$$

It should be noted that the tensor $\gamma_{ij}(0,\tau)$ is *diagonal.*

The expression (3.1) for the temporal degree of coherence of blackbody radiation is identical with the expression derived by Kano and Wolf[2] and discussed in detail by them [cf. Eq. (11) of their paper]. In particular they gave diagrams showing the behavior of the modulus and the argument of $\gamma_{ii}(0,\tau)$ as function of τ.

4. SPATIAL COHERENCE

Setting $\tau=0$ in Eq. (2.7) we obtain the following expression for the normalized tensor that characterizes *spatial coherence*:

$$\gamma_{ij}(\mathbf{r},0)=\frac{90}{\pi^4}\sum_{n=1}^{\infty}\left\{\frac{\delta_{ij}}{(n^2+r^2/\alpha^2)^2}+\frac{2}{\alpha^2}\frac{r_ir_j-\delta_{ij}r^2}{(n^2+r^2/\alpha^2)^3}\right\}. \quad (4.1)$$

It is seen from this expression that $\gamma_{ij}(\mathbf{r},0)$ is real. The series on the right-hand side of (4.1) may easily be summed with the help of the result[15]

$$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}=\frac{\pi}{2a}L(\pi a), \quad (4.2)$$

where $L(x)$ is the Langevin function

$$L(x)=\coth x-1/x. \quad (4.3)$$

Successive term by term differentiation of (4.2) gives

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+a^2)^2}=-\frac{1}{2a}\frac{d}{da}\left\{\frac{\pi}{2a}L(\pi a)\right\}, \quad (4.4)$$

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+a^2)^3}=\frac{\pi}{16a}\frac{d}{da}\left\{\frac{1}{a}\frac{d}{da}\left[\frac{1}{a}L(\pi a)\right]\right\}. \quad (4.5)$$

If we set

$$\mathbf{r}=(\alpha/\pi)\mathbf{r}', \quad (4.6)$$

so that r' represents the separation of points in the cavity in units of α/π, (4.1) gives, with the help of (4.4), (4.5), and (4.3),

$$\gamma_{ij}(\mathbf{r}')\equiv\gamma_{ij}(\mathbf{r},0)=\frac{45}{4r'^4}\left[A(r')\delta_{ij}+B(r')\frac{r_i'r_j'}{r'^2}\right], \quad (4.7)$$

where

$$\left.\begin{aligned}A(r')&=-r'\coth r'-r'^2\operatorname{csch}^2r'\\&\qquad-2r'^3\operatorname{csch}^2r'\coth r'+4,\\B(r')&=3r'\coth r'+3r'^2\operatorname{csch}^2r'\\&\qquad+2r'^3\operatorname{csch}^2r'\coth r'-8.\end{aligned}\right\} \quad (4.8)$$

[15] L. B. W. Jolley, *Summation of Series* (Dover Publications, New York, 1961), p. 22, Series number 124.

Equation (4.7) is valid for all $\mathbf{r}'$ but is not suitable for computing $\gamma_{ij}(\mathbf{r}')$ when r' is small. For small r', however, we directly obtain from (4.1), on expanding in powers of r',

$$\gamma_{ij}(\mathbf{r}')\equiv\gamma_{ij}(\mathbf{r},0)=\delta_{ij}\left[1-\frac{8}{21}r'^2+\frac{3}{35}r'^4+\cdots\right]+\frac{r_i'r_j'}{r'^2}\left[\frac{4}{21}r'^2-\frac{2}{35}r'^4+\cdots\right] \quad (r'<\pi). \quad (4.9)$$

If we set $i=j=x$ in (4.7) and (4.9) and take $\mathbf{r}'$ in the direction of the x axis, we obtain the following expressions for the *longitudinal coherence function* $\gamma_{xx}^{\text{long}}$:

$$\gamma_{xx}^{\text{long}}(\mathbf{r}')=\frac{45}{2r'^4}(r'\coth r'+r'^2\operatorname{csch}^2r'-2) \quad (r'<\infty), \quad (4.10\text{a})$$

$$=1-\frac{4}{21}r'^2+\frac{1}{35}r'^4+\cdots \quad (r'<\pi), \quad (4.10\text{b})$$

where $r'=x'=\pi x/\alpha$.

On the other hand, if $\mathbf{r}'$ is chosen perpendicular to the x axis, then we obtain from (4.7) the following expression for *lateral coherence function* γ_{xx}^{lat}:

$$\gamma_{xx}^{\text{lat}}(\mathbf{r}')=\frac{45}{4r'^4}[4-r'\coth r'-r'^2\operatorname{csch}^2r'-2r'^3\coth r'\operatorname{csch}^2r'], \quad (r'<\infty), \quad (4.11\text{a})$$

$$=1-\frac{8}{21}r'^2+\frac{3}{35}r'^4+\cdots \quad (r'<\pi), \quad (4.11\text{b})$$

where $r'=(y'^2+z'^2)^{1/2}=\pi(y^2+z^2)^{1/2}/\alpha$.

Expressions for $\gamma_{yy}^{\text{long}}$, γ_{yy}^{lat}, $\gamma_{zz}^{\text{long}}$, and γ_{zz}^{lat} are, of course, strictly similar. The expressions (4.10a)–(4.11b) are in agreement with those derived by Bourret.[1]

Next we will examine the three-dimensional distribution of $\gamma_{ij}(\mathbf{r}')$. First, let us consider a particular nondiagonal component, say the xy component. From (4.7) and (4.9), we obtain

$$\gamma_{xy}(\mathbf{r}')=\frac{45x'y'}{4r'^6}[3r'\coth r'+3r'^2\operatorname{csch}^2r'+2r'^3\operatorname{csch}^2r'\coth r'-8] \quad (r'<\infty), \quad (4.12\text{a})$$

$$\sim\frac{4}{21}x'y'+\cdots \quad (r'<\pi). \quad (4.12\text{b})$$

In Fig. 1, the xy section of the surface of constant $\gamma_{xy}(\mathbf{r}')$ is shown. $\gamma_{xy}(\mathbf{r}')$ is positive in the I and III quadrants and negative in II and IV quadrants, attaining peak values at four points, denoted by letters A_1, A_2, B_1, B_2 in the figure, at distances $r=\alpha r'/\pi\sim2.3\alpha/\pi$ from

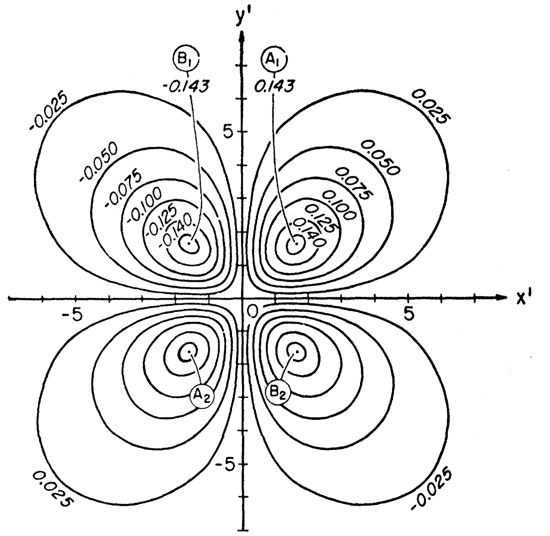

FIG. 1. Contours of $\gamma_{xy}(\mathbf{r}')$ in the xy plane. $\gamma_{xy}(\mathbf{r}')\equiv\gamma_{xy}(\mathbf{r},0) = (1/N)\,\mathcal{E}_{xy}(\mathbf{r},0)$; $N=\mathcal{E}_{xx}(0,0)=(64/45)(\pi^2/hc)^3K^4T^4$; $r=\alpha r'/\pi$; $\alpha=\hbar c/KT$.

the origin, this distance being of the order of half the mean wavelength of the radiation.[4] The variation of $\gamma_{xy}(\mathbf{r}')$ along the line $x=y$, $z=0$, is shown in Fig. 2. In Fig. 3, the contours of the surfaces $\gamma_{xy}(\mathbf{r}')$ in the plane $x=y$ are shown.

We see that the surfaces, $\gamma_{xy}(\mathbf{r}')=\text{constant}$, are closed surfaces, having the lines $x=\pm y$, $z=0$ as axes of symmetry. (Thus, any plane section through one of these lines will be symmetrical about that line.) Near the origin these surfaces are more or less hyperbolic cylinders as seen from Eq. (4.12b). The surfaces for peak values $|\gamma_{xy}(\mathbf{r}')|\simeq 0.143$, shrink to the four points A_1, A_2, B_1, B_2. The behavior of the other nondiagonal components, such as $\gamma_{yz}(\mathbf{r}')$ or $\gamma_{zx}(\mathbf{r}')$ is, of course, strictly similar.

For diagonal components of the correlation tensor, it is again sufficient to consider one of them, e.g., that

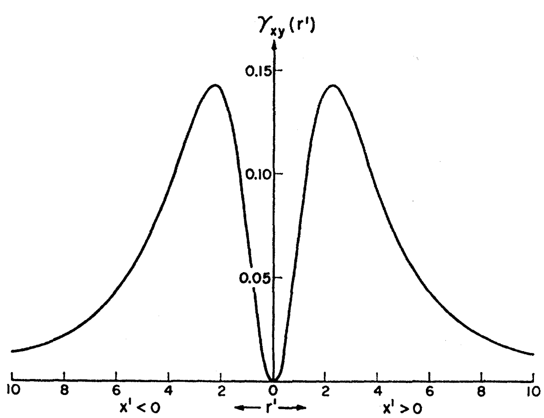

FIG. 2. Variation of $\gamma_{xy}(\mathbf{r}')$ with $\mathbf{r}'$, when $\mathbf{r}'$ is in the direction $x=y$ in the plane $z=0$.

corresponding to $i=j=z$. We have from (4.7) and (4.9)

$$\gamma_{zz}(r')=\frac{45}{4r'^4}\Big\{\big(-r'\coth r'-r'^2\operatorname{csch}^2 r' -2r'^3\coth r'\operatorname{csch}^2 r'+4\big)+\frac{z'^2}{r'^2}\big(3r'\coth r' +3r'^2\operatorname{csch}^2 r'+2r'^3\coth r'\operatorname{csch}^2 r'-8\big)\Big\}, \quad (r'<\infty), \qquad (4.13a)$$

$$=1-\frac{8}{21}r'^2+\frac{4}{21}z'^2+\cdots \quad (r'<\pi). \qquad (4.13b)$$

Eq. (4.13b) shows that for small spatial separation r' of the two points, the surfaces $\gamma_{zz}(\mathbf{r}')=\text{constant}$ are the ellipsoids

$$2(x^2+y^2)+z^2=\text{constant}.$$

The contours of $\gamma_{zz}(\mathbf{r}')$ in the xy plane are circles, shown in Fig. 4. Figure 5 shows the contours in the

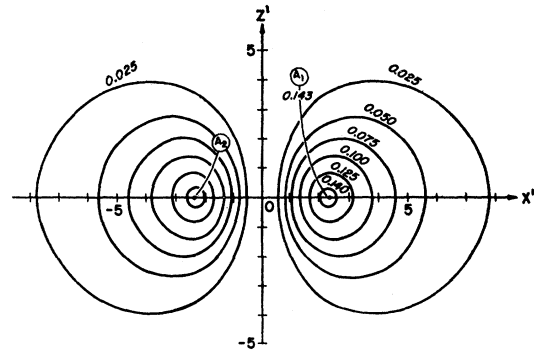

FIG. 3. Contours of $\gamma_{xy}(\mathbf{r}')$ in the plane $x=y$.

yz plane. The actual surfaces $\gamma_{zz}(\mathbf{r}')=\text{constant}$ are just the surfaces of revolution generated by rotating these contours (Fig. 5) about the z axis.

For the sake of completeness the variation of γ_{zz} along the z axis (longitudinal coherence) and along the x axis (lateral coherence) are given in Figs. 6 and 7, respectively. These two curves are in agreement with Figs. 3 and 4 of Bourret.[1]

In the present section we have discussed only the case of spatial coherence characterized by $\gamma_{ij}(\mathbf{r}')\equiv\gamma_{ij}(\mathbf{r},0)$. In the general case, when τ is arbitrary, i.e., when the correlation is characterized by $\gamma_{ij}(\mathbf{r},\tau)$ rather than by $\gamma_{ij}(\mathbf{r},0)$, the value of the correlation, for any particular separation r of the two points may, of course, become appreciably larger (or smaller), provided τ is chosen appropriately.

5. THE SECOND-ORDER COMPLEX MAGNETIC AND MIXED CORRELATION TENSORS OF BLACKBODY RADIATION

Besides coherence effects associated with the electric field, coherence effects involving the magnetic field are also of interest. Some of these effects are characterized by a magnetic correlation tensor $\mathcal{H}_{ij}$ and mixed correlation tensors $\mathcal{G}_{ij}$ and $\tilde{\mathcal{G}}_{ij}$, introduced in earlier publications.[16a,b] These tensors may be defined in a similar manner as the electric correlation tensor $\mathcal{E}_{ij}$. Assuming again that the field is stationary we have, in analogy with (1.3a),

$$\mathcal{H}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\langle H_i(\mathbf{r}_1,t+\tau)H_j^*(\mathbf{r}_2,t)\rangle\,, \qquad (5.1)$$

$$\mathcal{G}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\langle E_i(\mathbf{r}_1,t+\tau)H_j^*(\mathbf{r}_2,t)\rangle\,, \qquad (5.2)$$

$$\tilde{\mathcal{G}}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\langle H_i(\mathbf{r}_1,t+\tau)E_j^*(\mathbf{r}_2,t)\rangle\,, \qquad (5.3)$$

where, as before $\mathbf{E}(\mathbf{r},t)$ denotes the complex analytic signal associated with the real electric field $\mathbf{E}^{(r)}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ denotes the complex analytic signal associated with the real magnetic field $\mathbf{H}^{(r)}(\mathbf{r},t)$.

We note that the tensors $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are related as follows:

$$\tilde{\mathcal{G}}_{ij}(\mathbf{r}_1,\mathbf{r}_2,\tau)=\mathcal{G}_{ji}^*(\mathbf{r}_2,\,\mathbf{r}_1,\,-\tau)\,. \qquad (5.4)$$

It is known[16a,17] that *in vacuo* each of these tensors

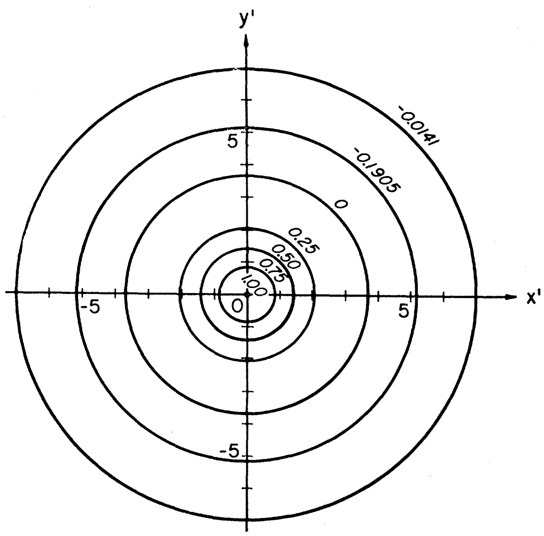

FIG. 4. Contours of $\gamma_{zz}(\mathbf{r}')$ in the xy plane.

satisfies two homogeneous wave equations. Also, that the tensors are not independent but satisfy differential equations of the form[16b,17]

$$\epsilon_{jkl}\partial_k{}^1\mathcal{E}_{lm}+\frac{1}{c}\frac{\partial}{\partial\tau}\tilde{\mathcal{G}}_{jm}=0\,, \qquad (5.5)$$

$$\epsilon_{jkl}\partial_k{}^1\mathcal{G}_{lm}+\frac{1}{c}\frac{\partial}{\partial\tau}\mathcal{H}_{jm}=0\,, \qquad (5.6)$$

[16] E. Wolf, (a) Nuovo Cimento **12**, 884 (1954); (b) in *Proceedings of the Symposium on Astronomical Optics*, edited by Z. Kopal (North-Holland Publishing Company, Amsterdam 1956), p. 177.
[17] P. Roman and E. Wolf, Nuovo Cimento **17**, 462 (1960).

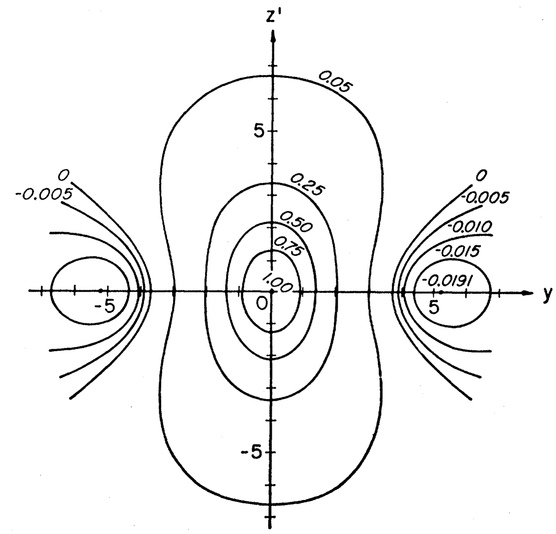

FIG. 5. Contours of $\gamma_{zz}(\mathbf{r}')$ in the yz plane. (Surfaces of constant γ_{zz} are the surfaces generated by rotation of these curves about the z axis).

etc., where $\partial_k{}^1$ ($k=x,\,y,\,z$) are the components of the gradient, taken with respect to the coordinates of $\mathbf{r}_1$ and ϵ_{jkl} is the completely antisymmetric unit tensor of Levi-Civita. There is a similar set of differential equations involving the components $\partial_k{}^2$ of the gradient, taken with respect to the coordinates of $\mathbf{r}_2$. In the case of blackbody radiation, one has, on account of isotropy $\partial_k{}^1=-\partial_k{}^2=\partial/\partial r_k$ where r_k are components of the vector $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$.

Now for blackbody radiation we have from (2.1)–(2.4)

$$\mathcal{E}_{ij}(\mathbf{r},\tau)=\frac{hc}{2\pi^3}\int\frac{k^2\delta_{ij}-k_ik_j}{k\{\exp(\alpha k)-1\}}\times\exp\{i(\mathbf{k}\cdot\mathbf{r}-kc\tau)\}d^3k\,. \qquad (5.7)$$

On substituting from (5.7) into (5.5) and solving the resulting equation for $\tilde{\mathcal{G}}$, subject to the boundary condi-

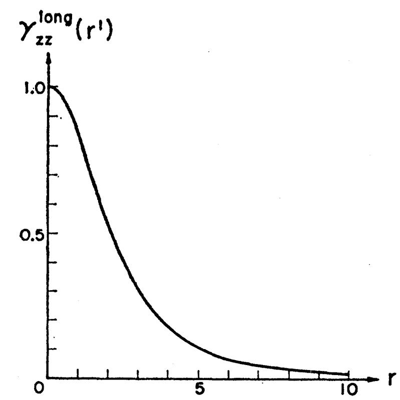

FIG. 6. Longitudinal coherence. Variation of $\gamma_{zz}{}^{\text{long}}(\mathbf{r}')\equiv\gamma_{zz}(\mathbf{r},0)$, with r', when $\mathbf{r}'$ is along the z axis.

tion $\tilde{\mathcal{G}}_{jm}(\mathbf{r},\infty)=0$, we obtain, if we also use (5.4),

$$\tilde{\mathcal{G}}_{jm}(\mathbf{r},\tau)=-\mathcal{G}_{jm}(\mathbf{r},\tau)=-\frac{hc}{2\pi^3}\epsilon_{jmk}\int\frac{k_k}{\exp(\alpha k)-1}\times\exp\{i(\mathbf{k}\cdot\mathbf{r}-kc\tau)\}d^3k\,. \quad (5.8)$$

From (5.8) and (5.6) we find on solving for $\mathcal{H}_{jm}$ subject to the boundary condition $\mathcal{H}_{lm}(\mathbf{r},\infty)=0$, and on comparing the resulting expression with (5.7),

$$\mathcal{H}_{jm}(\mathbf{r},\tau)\equiv\mathcal{E}_{jm}(\mathbf{r},\tau)\,. \quad (5.9)$$

In deriving (5.9) use has been made of the identity[18] $\epsilon_{jkl}\epsilon_{mk'j}=\delta_{km}\delta_{lk'}-\delta_{kk'}\delta_{lm}$. Relation (5.9) implies that *the magnetic coherency tensor is identical with the electric coherency tensor,* discussed in detail in the preceeding sections.

Let us now consider the mixed tensors $\mathcal{G}$ and $\tilde{\mathcal{G}}$. From (5.8) we may readily derive series expansions for these tensors. For this purpose, we first rewrite (5.8) in the form

$$\mathcal{G}_{jm}(\mathbf{r},\tau)=-\tilde{\mathcal{G}}_{jm}(\mathbf{r},\tau)=\frac{hc}{2\pi^3}\epsilon_{jmk}(-i\partial_k)\int\frac{\exp\{i(\mathbf{k}\cdot\mathbf{r}-kc\tau)\}}{\exp(\alpha k)-1}d^3k \quad (5.10)$$

and apply to the integral on the right-hand side of (5.10) a similar procedure as used in connection with Eq. (2.5). We then obtain

$$\mathcal{G}_{jm}(\mathbf{r},\tau)=-\tilde{\mathcal{G}}_{jm}(\mathbf{r},\tau)=i\frac{16hc}{\pi^2}\epsilon_{jmk}r_k\times\sum_{n=1}^{\infty}\frac{n\alpha+ic\tau}{[(n\alpha+ic\tau)^2+r^2]^3}\,. \quad (5.11)$$

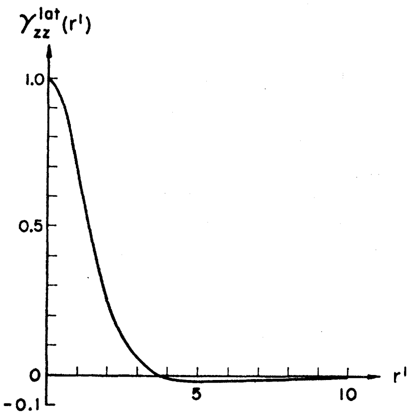

FIG. 7. Lateral coherence. Variation of $\gamma_{zz}^{lat}(\mathbf{r}')\equiv\gamma_{zz}(\mathbf{r},0)$, with r', when $\mathbf{r}'$ is along the x axis.

[18] H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics* (Cambridge University Press, New York, 1950), 2nd ed., p. 73.

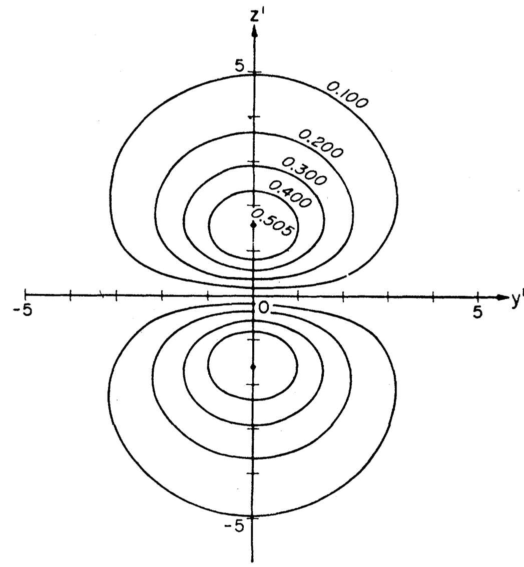

FIG. 8. Contours of $|\sigma_{xy}(\mathbf{r}')|\equiv|\sigma_{xy}(\mathbf{r},0)|$ in the yz plane. $\sigma_{xy}(\mathbf{r}')\equiv\sigma_{xy}(\mathbf{r},0)=(1/N)\mathcal{G}_{xy}(\mathbf{r},0)$; $N=(64/45)(\pi^2/hc)^3K^4T^4$.

We see that *the mixed tensors* $\mathcal{G}_{jm}$ and $\tilde{\mathcal{G}}_{jm}$ *are antisymmetric* and that

$$\mathcal{G}_{jm}(0,\tau)=\tilde{\mathcal{G}}_{jm}(0,\tau)=0\,. \quad (5.12)$$

Equation (5.12) implies that at every point $\mathbf{r}$, $E_i(\mathbf{r},t+\tau)$ and $H_j^*(\mathbf{r},t)$ [and also, of course $H_i(\mathbf{r},t+\tau)$ and $E_j^*(\mathbf{r},t)$] are *uncorrelated* irrespective of the time delay τ, i.e., *there is no "temporal coherence" between the complex electric and the complex magnetic field.*

It will be convenient to normalize the tensors $\mathcal{G}$ and $\tilde{\mathcal{G}}$ in a similar manner as we normalized $\mathcal{E}$, i.e., we define normalized mixed correlation tensors σ and $\tilde{\sigma}$ by the formulas

$$\sigma_{jm}(\mathbf{r},\tau)=\frac{1}{N}\mathcal{G}_{jm}(\mathbf{r},\tau)\,,\quad \tilde{\sigma}_{jm}(\mathbf{r},\tau)=\frac{1}{N}\tilde{\mathcal{G}}_{jm}(\mathbf{r},\tau)\,, \quad (5.13)$$

where [cf. (2.1), (2.4), and (5.9)]

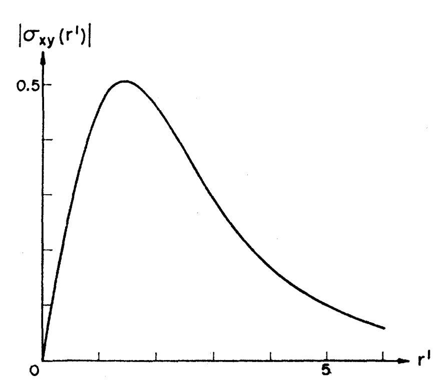

FIG. 9. Variation of $|\sigma_{xy}(\mathbf{r}')|$ with r', when $\mathbf{r}'$ is along the z axis.

$$N=[\mathcal{E}_{jj}(0,0)]^{1/2}[\mathcal{H}_{jj}(0,0)]^{1/2}$$
$$=\mathcal{E}_{jj}(0,0)=\mathcal{H}_{jj}(0,0)=\frac{64}{45}\frac{\pi^6K^4T^4}{(hc)^3}. \quad (5.14)$$

It follows from (5.13), (5.14), and (5.11) that σ and $\tilde{\sigma}$ may be represented in series form as follows:

$$\sigma_{jm}(\mathbf{r},\tau)=-\tilde{\sigma}_{jm}(\mathbf{r},\tau)=i\frac{180\alpha^4}{\pi^4}r_k\epsilon_{jmk}$$
$$\times\sum_{n=1}^{\infty}\frac{n\alpha+ic\tau}{[(n\alpha+ic\tau)^2+r^2]^3}. \quad (5.15)$$

Here the constant α is given by (2.3) as before.

From (5.15) we obtain in the special case $\tau=0$ (*spatial coherence* between the electric and magnetic fields), for typical nondiagonal elements of σ and $\tilde{\sigma}$,

$$\sigma_{xy}(\mathbf{r},0)=-\tilde{\sigma}_{xy}(\mathbf{r},0)=i\frac{180}{\alpha\pi^4}z\sum_{n=1}^{\infty}\frac{n}{(n^2+r^2/\alpha^2)^3}. \quad (5.16)$$

The diagonal elements are, of course, identically zero, since σ is antisymmetric.

We see from (5.16) that in the plane $z=0$ (xy plane) $\sigma_{xy}(\mathbf{r},0)$ is identically zero. In Fig. 8 the contours of $|\sigma_{xy}(\mathbf{r},0)|$ in the yz plane are presented and in Fig. 9 the variation of $|\sigma_{xy}(\mathbf{r},0)|$ along the z axis is shown.

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Coherence Properties of Blackbody Radiation.* II. Correlation Tensors of the Quantized Field

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Expressions are derived for the electromagnetic correlation tensors of blackbody radiation defined on the basis of the theory of the quantized field. Correlation functions of all order are considered, but second-order ones are discussed in detail; it is found that these are identical with those obtained on the basis of semiclassical theory in part I of this investigation. This result illustrates a recent theorem of E.C.G. Sudarshan relating to the equivalence between semiclassical and quantum mechanical description of statistical light beams.

1. INTRODUCTION

IN part I of this investigation,[1] expressions were derived for the complex second-order electromagnetic correlation tensors of blackbody radiation and their behavior was discussed in detail and illustrated by a number of diagrams. The statistical methods used were based entirely on classical concepts, though quantum mechanical features of the radiation were implicit in that treatment, since the spectrum of the radiation was taken to be given by Planck's law.

In the present paper the second-order correlation tensors introduced recently by Glauber[2] on the basis of the theory of the quantized field, are evaluated for blackbody radiation and are shown to be identical with those defined and evaluated on the basis of the semiclassical theory. This result illustrates a recent theorem of Sudarshan,[3] relating to the equivalence between semiclassical[4–6] and quantum mechanical description of statistical light beams.

In Sec. 4 higher-order correlation tensors of blackbody radiation are briefly discussed.

2. THE SECOND-ORDER CORRELATION TENSORS OF THE QUANTIZED FIELD

It will be useful to begin with some results which will be needed later, relating to quantization of the electromagnetic field and the definition of the correlation tensors of the quantized field.

The electric-field operator, at the space-time point $x\equiv(\mathbf{r},ct)$, when expanded in a Fourier series is given by[7] (with $i=x, y, z$)

$$\hat{E}_i(x)=\hat{E}_i^{(+)}(x)+\hat{E}_i^{(-)}(x), \quad (2.1)$$

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[1] C. L. Mehta and E. Wolf, Phys. Rev. **134A**, 1143 (1964), preceding paper.

[2] R. J. Glauber, (a) *Electronique Quantique, 3eme Conference*, edited by N. Bloembergen and P. Grivet (Dunod Cie, Paris, 1964), p. 111; (b) Phys. Rev. **130**, 2529 (1963); (c) *ibid.* **131**, 2766 (1963).

[3] E. C. G. Sudarshan, (a) Phys. Rev. Letters **10**, 277 (1963). (b) in *Proceedings of the Symposium on Optical Masers* (Polytechnique Press, Brooklyn, New York and John Wiley & Sons, Inc., New York, 1963), p. 45.

[4] The term "semiclassical" implies here that the distribution functions characterizing statistical properties of the beam are not necessarily non-negative and may therefore not be true probabilities. They are essentially Wigner distribution functions (see Refs. 5 and 6), called also "quasiprobabilities." However, in the present case (blackbody radiation) the distribution function turns out to be positive. [See Eq. (3.1).]

[5] E. P. Wigner, Phys. Rev. **40**, 749 (1932).

[6] (a) J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1949). (b) G. A. Baker, Jr., Phys. Rev. **109**, 2198 (1958). (c) C. L. Mehta, J. Math. Phys. **5**, 677 (1964).

[7] All operators are denoted by circumflex.